Analytical signal formalism in the description of optical pulse photodetection *

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Abstract

In this paper, we unveil and discuss a theoretical misconception in a classical formalism of Gaussian pulse envelope propagation. With simple analytical arguments, we explain the mistake and develop a universal result, valid for short and ultra-short optical pulses of arbitrary shape. This general result sheds light on the mathematical separation of a pulse into its envelope (energy) and its coherent characteristics (frequency chirp). With it, a theoretical ground is provided for the use of blind numerical computations to calculate complex photodetected pulse-shapes.

Key terms: ultrashort optical pulses, envelope photodetection, analytical signal formalism, linear fiber propagation, non-narrowband signals.

# 1 Introduction

The theory of linear pulse propagation in single-mode fibers has provided well-established, widely known results about the effects of chromatic dispersion on the pulse shape and phase, and, ultimately, on the maximum transmission rate of digital signals propagated through optical fibers (see for example \[1\]). With the remarkable and prototypical exception of a Gaussian pulse, the actual computation of pulse propagation relies on numerical methods (typically, Fourier transform computations by the FFT algorithm), as analytical solutions are generally not available.

A pulse-modulated optical carrier transmitted through a fiber is a clear example of a bandpass signal where the magnitude of the carrier frequency is enormous as compared with the spectral width of the modulating pulse train; this is so even for ultrashort pulses in the picosecond or sub-picosecond (hundreds of femtoseconds) range (\[2\] for a comprehensive treatment). In such a situation, it is only the pulse envelope that can be dealt with numerically; the optical frequency cannot—and need not—be tracked with the FFT or any such method. However, the dispersive nature of the propagation generally induces additional frequency modulation in the intensity-modulated light that impinges on the detector. This poses certain difficulties to the separation of the genuine envelope which would be retrieved by an optical photodetector working in the standard, direct-detection mode. Other features of the received electric field which physically pass unnoticed by the detector may mathematically arise when a numerical, “blind” approach is used.

To illustrate the point above, we shall use, in the next section, the well-known case of a Gaussian pulse, for which simple analytical results exist when the propagation constant $\beta(\omega)$ is approximated to second order in $\omega$. We shall see that a misguided treatment of the problem in the literature may bring about certain erroneous results. In section 4 we develop the theoretical formalism that allows for the generalization of the results for Gaussian pulses, obtained in section 2, to arbitrary pulses of any shape. Previously, a very brief review of the analytical signal theory is given in section 3.
2 Direct photodetection of a propagated Gaussian pulse: fictitious “extra-oscillations”

As announced in the introduction, we shall approach our discussion by first considering the specific and simple example of a Gaussian pulse. Ignoring the modal profile and other irrelevant constants, the linearly-polarized electric field at the input of a single-mode fiber is given by

$$E(z = 0; t) = e^{-t^2/2T^2} \cos(\omega_0 t) \quad (1)$$

when the optical power emitted by the ideally monochromatic source ($\omega_0$) is modulated as $\sim \exp(-t^2/T^2)$.

To proceed with the Fourier analysis, one works only with $\exp(i\omega_0 t)$, which amounts to choosing the positive-frequency part of $\cos(\omega_0 t)$ ($+\omega_0$). For the moment, we shall loosely call this function the “analytical field,” which will be denoted with a tilde: $\tilde{E}(0; t)$. As we shall point out later, $\tilde{E}(0; t)$ is a good approximation to, but does not strictly coincide with, the true analytical signal. We thus write

$$\tilde{E}(0; t) = e^{-t^2/2T^2} e^{i\omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\Omega) e^{i(\omega_0 + \Omega)t} d\Omega, \quad (2)$$

with $F(\Omega)$ the Fourier transform (FT) of $\exp(-t^2/2T^2)$; namely,

$$F(\Omega) = \sqrt{2\pi} \exp(-\Omega^2 T^2/2).$$

The propagated field will be

$$\tilde{E}(z; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\Omega) e^{i[\omega_0 + \Omega]t - \beta(\omega_0 + \Omega)z} d\Omega. \quad (3)$$

If the propagation constant $\beta$ is Taylor-expanded to second order in $\Omega$ ($\beta(\omega_0 + \Omega) \simeq \beta_0 + \Omega/v_g + \beta_2 \Omega^2/2$, with $v_g$ the group velocity at $\omega_0$), the integral in (3) is solvable, yielding

$$\tilde{E}(z; t) = \frac{T}{\sqrt{T^2 + i\beta z}} \exp \left[ - \frac{T^2(t - z/v_g)^2}{2(T^4 + \beta_2 z^2)} \right] \times \exp \left[ i \left( \omega_0 t - \beta_0 z + \frac{\beta_2 z (t - z/v_g)^2}{2(T^4 + \beta_2 z^2)} \right) \right]. \quad (4)$$

The actual real-valued, time-varying electric field is the real part of $\tilde{E}(z; t)$:

$$E(z; t) = \text{Re} \left[ \tilde{E}(z; t) \right] = \frac{T}{\sqrt{T^4 + \beta_2 z^2}} \exp \left[ - \frac{T^2(t - z/v_g)^2}{2(T^4 + \beta_2 z^2)} \right]$$

$$\times \cos \left[ \omega_0 t - \beta_0 z + \frac{\beta_2 z (t - z/v_g)^2}{2(T^4 + \beta_2 z^2)} - \frac{1}{2} \log \left( \frac{\beta_2 z}{T^2} \right) \right]. \quad (5)$$
Expression (5) is a very well known, almost trivial result, yet a number of misunderstandings may be generated from a careless analysis of it — analyses in the literature tend to start from the form (4) rather than the more “genuine” (5); this probably increases the risk of mistake. First, the argument of the cosine contains the optical phase, $\omega_0 t$, as a summand, even if chirped with a linear frequency modulation caused by the fiber dispersion ($\beta_2$). Therefore, the presence of $\omega_0$ necessarily makes the cosine term transparent to any quadratic photodetector working in direct (non-coherent) reception, regardless of the more or less complex additional frequency modulation. It is only the square of the real envelope in (5) that matters, as this is the magnitude actually reproduced by the photocurrent:

$$a(z; t) \equiv \frac{T^2}{\sqrt{T^4 + \beta_2^2 z^2}} \exp \left[ -\frac{T^2(t - z/v_g)^2}{(T^4 + \beta_2^2 z^2)} \right]. \quad (6)$$

However, a remarkable misconception can be found in [3]. In this notorious reference, numerical calculations are performed for various cases of Gaussian pulse propagation (including dispersion up to second order, $\beta_3$, and a non-monochromatic source). Surprisingly, the result obtained for the case corresponding to the one considered here shows an instantaneous pulse power as depicted in figure 1, where fast (but not optical) oscillations unexpectedly appear within the true pulse shape, as given by (6).

The analytical justification given in [3] to explain the (actually unreal) ringing pulse is as follows: Starting from an expression like (4), the effect of the optical carrier ($\omega_0$) “is removed by multiplying the function with $\exp(-i\omega_0 t)$.” The square of the real part of this product is what is represented in solid line in figure 1. Of course, what such a procedure yields is

$$\left\{ \text{Re} \left[ \tilde{E}(z; t) e^{-i\omega_0 t} \right] \right\}^2 \equiv \psi^2(z; t) = \frac{T^2}{\sqrt{T^4 + \beta_2^2 z^2}} \exp \left[ -\frac{T^2(t - z/v_g)^2}{(T^4 + \beta_2^2 z^2)} \right] \times \cos^2 \left[ \frac{\beta_2 z (t - z/v_g)^2}{2(T^4 + \beta_2^2 z^2)} - \frac{1}{2} \text{sgn} \left( \frac{\beta_2 z}{T^2} \right) \right], \quad (7)$$

so that the fictitious oscillations arise from the squared cosine in (7). A “receiver with a narrow electrical bandwidth” is suggested in [3] as an ad hoc remedy that would suppress the hypothetical extra-oscillations. Naturally, no such extra-oscillations really exist because the derivation is mathematically inconsistent in the first place; the correct power pulse is just given by (6).

Such theoretical error in the pioneering work [3] deserves more consideration that it might seem
at first sight, because its study will shed light on certain aspects relevant to the general theory of incoherent detection of ultrashort fiber-propagated pulses. First, we note that the ultimate origin of the misunderstanding discussed above lies, to a great extent, on the fact that a great deal of numerical computations were carried out in [3]. As we anticipated in the introduction, careless numerical handling of optical pulses is prone to missing the true pulse envelope. Let us see this for the case of the Gaussian pulse.

Firstly, the optical carrier needs to be removed from the calculations. We write

$$\beta(\omega_0 + \Omega) = \beta_0 + \Delta \beta,$$

so (3) becomes

$$\tilde{E}(z; t) = e^{i(\omega_0 t - \beta_0 z)}\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\Omega t} e^{-i \Delta \beta z} d\Omega = e^{i(\omega_0 t - \beta_0 z)}\text{FT}^{-1} [e^{-i \Delta \beta z}].$$

(8)

On the other hand, we know that (with $\beta$ approximated to second order)

$$\tilde{E}(z; t) = \frac{T}{\sqrt{T^2 + i\beta_2 z}} \exp \left[ -\frac{T^2 (t - z/v_g)^2}{2(T^4 + \beta_2^2 z^2)} \right] \times \exp \left\{ i \left[ \omega_0 t - \beta_0 z + \frac{\beta_2 z (t - z/v_g)^2}{2(T^4 + \beta_2^2 z^2)} \right] \right\}. \tag{9}$$

Comparing (8) and (9), the immediate result (10) follows:

$$\text{FT}^{-1} [e^{-i \Delta \beta z}] = \frac{T}{\sqrt{T^2 + i\beta_2 z}} \exp \left[ -\frac{T^2 (t - z/v_g)^2}{2(T^4 + \beta_2^2 z^2)} \right] \times \exp \left\{ \frac{\beta_2 z (t - z/v_g)^2}{2(T^4 + \beta_2^2 z^2)} \right\}. \tag{10}$$

Therefore, if we compute $\text{FT}^{-1}[\exp(-i \Delta \beta z)]$, for example with the FFT algorithm, and take its real part—as in [3]—, we shall obtain precisely $\psi(z; t)$, which does not correspond to the true shape of the electric field envelope. In order to obtain the correct result, we see that it is the modulus, and not the real part, of $\text{FT}^{-1}[\exp(-i \Delta \beta z)]$ that should be taken.

The conclusion above applies only for a Gaussian pulse, and has been easily obtained thanks to the fact that an analytical solution, that can be used for comparison, exists for a Gaussian pulse. The question in now obvious: Since, in general, we shall have no analytical solution or clue to guide us, can our result be generalized to optical pulses of arbitrary shape? Intuitively, one expects that to be the case; but no formal proof of it has yet been given, to the best of our knowledge. We shall provide such proof in section 4, but we first need to briefly review a number of concepts of the analytical signal theory, which we undertake next.
3 Review of the Analytical Signal concept for bandpass signals

In general, if $F(t)$ is an arbitrary (not necessarily bandpass) real signal, its analytical signal, which we shall denote as $\tilde{F}(t)$, is defined as the function whose Fourier transform is as follows:

$$ \tilde{F}(\omega) = \begin{cases} 2F(\omega), & \omega > 0 \\ F(0), & \omega = 0 \\ 0, & \omega < 0 \end{cases} \quad (11) $$

It can be shown that $\tilde{F}(t)$ may be written in the form (see for example, [4], [5])

$$ \tilde{F}(t) = F(t) + i\tilde{F}(t), \quad (12) $$

where $\tilde{F}(t)$ is the Hilbert transform of $F(t)$. The prototypical and simplest example is the complex phasor at frequency $\omega_0$:

$$ \exp(i\omega_0 t) = \cos(\omega_0 t) + i\sin(\omega_0 t). \quad (13) $$

A number of relationships can be easily derived involving $\tilde{F}(t)$, $F(t)$, $\tilde{F}(t)$ and the so-called negative- and positive-frequency parts, $F^-(t)$ and $F^+(t)$:

$$ F(t) = F^-(t) + F^+(t) \quad (14) $$

$$ \tilde{F}(t) = i[F^-(t) - F^+(t)] \quad (15) $$

$$ \tilde{F}(t) = 2F^+(t). \quad (16) $$

The negative- and positive-frequency parts are defined as those (complex) time-functions having only negative and positive frequencies, respectively:

$$ F^-(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{0} F(\omega) e^{i\omega t} d\omega, \quad F^+(t) \equiv \frac{1}{2\pi} \int_{0}^{\infty} F(\omega) e^{i\omega t} d\omega \quad (17) $$

(From (17), the verification of relationships (14)--(16) is obvious.)

To finish, we also need to introduce the so-called complex envelope (or pre-envelope), which will prove to be useful for our purposes. The complex envelope of $\tilde{F}(t)$, denoted $r(t)$, is defined as...
\[ r(t) = r(t) e^{i\alpha(t)} = \tilde{F}(t) e^{-i\omega_0 t}, \]  
where \( r(t) \) and \( \alpha(t) \) are the modulus and phase of \( r(t) \), respectively. The parameter \( \omega_0 \) is in principle arbitrary, but, as we shall soon see, the decomposition (18) will be useful only if \( \mathcal{F}(t) \) is a bandpass signal and \( \omega_0 \) is precisely chosen to be its carrier frequency.

The following result can be derived in any case:

\[ \mathcal{F}(t) = r(t) \cos[\omega_0 t + \alpha(t)]. \]  

We note in advance that, with the appropriate choice of \( \omega_0 \) as the optical carrier frequency, we have found in (19) a completely general expression of the type we are seeking, that is, of the type (5).

4 Real envelope for arbitrary pulses

We shall first present the expression of an arbitrary pulse propagated through a step-index, single-mode waveguide with a modal profile \( u(x, y, \omega) \) and a propagation constant \( \beta(\omega) \); the laser source is assumed chirpless and monochromatic of frequency \( \omega = \omega_0 \), and the waveguide index is \( n(\omega; x, y) \). Although this derivation might appear elemental and superfluous, we find it useful to rigorously establish a clear notation, which is critical to prevent any possible confusion between real-time field, analytical field, positive-frequency field, forward-propagating field, and so on.

Let the (real-time) electric field at \( z = 0 \) be

\[ \mathcal{E}(t; x, y, z = 0) = u(x, y, \omega_0) g(t) \cos(\omega_0 t), \]  
where \( g(t) \) is the pulse envelope. \( \mathcal{E}(t; x, y, z = 0) \) can be written in terms of its Fourier transform \( E(\omega, x, y, z = 0) \) in the usual way:

\[ \mathcal{E}(t; x, y, z = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega; x, y, 0) e^{i\omega t} d\omega. \]  
Calling \( G(\omega) \) the FT of \( g(t) \), the expression of \( E(\omega, x, y, z = 0) \) is

\[ E(\omega; x, y, z = 0) = u(x, y, \omega_0) \left\{ \frac{G(\omega - \omega_0)}{2} + \frac{G(\omega + \omega_0)}{2} \right\}. \]
The electric field \( E(\omega; x, y, z) \) is the solution of the \((\omega\text{-space})\) wave equation

\[
\nabla^2 E(\omega; x, y, z) + \left[ \frac{\omega}{c} n(\omega; x, y) \right]^2 E(\omega; x, y, z) = 0.
\]

Considering the forward-propagating wave only, the solution reads

\[
E(\omega; x, y, z) = E(\omega; x, y, z = 0) \exp[-i \beta(\omega) z] = u(x, y, \omega_0) \left\{ \frac{G(\omega - \omega_0)}{2} + \frac{G(\omega + \omega_0)}{2} \right\} \exp[-i \beta(\omega) z].
\]

The time-domain electric field at \( z \) will thus be

\[
\mathcal{E}(t; x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\omega; x, y, z) e^{i\omega t} d\omega = u(x, y, \omega_0) \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{G(\omega - \omega_0)}{2} + \frac{G(\omega + \omega_0)}{2} \right\} e^{i\omega t} e^{-i \beta(\omega) z} d\omega.
\]

For notational brevity, we shall omit the modal profile \( u(x, y, \omega_0) \) hereafter, since it plays no role in the following discussion. Thus,

\[
\mathcal{E}(t; z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{G(\omega - \omega_0)}{2} + \frac{G(\omega + \omega_0)}{2} \right\} e^{i\omega t} e^{-i \beta(\omega) z} d\omega. \tag{21}
\]

In general, almost any \( G(\omega) \) will have a bandwidth which is strictly infinite, so the left “tail” of \( G(\omega - \omega_0) \) will extend into the negative-frequency semi-axis (similarly, \( G(\omega + \omega_0) \) will have positive-frequency components). We shall write (21) as a sum of a quasi positive-frequency part, \( \mathcal{E}^{(+)}(t; x, y, z) \), and a quasi negative-frequency part, \( \mathcal{E}^{(-)}(t; x, y, z) \):

\[
\mathcal{E}(t; z) = \mathcal{E}^{(-)}(t; z) + \mathcal{E}^{(+)}(t; z) + \epsilon(t; z),
\]

where

\[
\mathcal{E}^{(-)}(t; z) = \frac{1}{2\pi} \int_{-\infty}^{0} \frac{G(\omega + \omega_0)}{2} e^{i\omega t} e^{-i \beta(\omega) z} d\omega, \quad \mathcal{E}^{(+)}(t; z) = \frac{1}{2\pi} \int_{0}^{\infty} \frac{G(\omega - \omega_0)}{2} e^{i\omega t} e^{-i \beta(\omega) z} d\omega,
\]

and the —usually negligible— “error” \( \epsilon(t; z) \) is given by
\[
e(t; z) = \frac{1}{2\pi} \int_{-\infty}^{0} \frac{G(\omega - \omega_0)}{2} e^{i\omega t} e^{-i\beta(\omega) z} \, d\omega + \frac{1}{2\pi} \int_{0}^{\infty} \frac{G(\omega + \omega_0)}{2} e^{i\omega t} e^{-i\beta(\omega) z} \, d\omega
\]

\[
\equiv e^{(-)}(t; z) + e^{(+))(t; z)};
\]

(22)

see figure 2.

We introduce the notation \(\mathcal{E}^{-}(t; z)\) (where the minus superscript is *not* in parentheses) to refer to the *exact* negative-frequency field, and similarly \(\mathcal{E}^{+}(t; z)\) for the positive part; i.e.,

\[
\mathcal{E}^{-}(t; z) = \mathcal{E}^{(-)}(t; z) + e^{(-)}(t; z)
\]

\[
\mathcal{E}^{+}(t; z) = \mathcal{E}^{(+))(t; z)} + e^{(+))(t; z)}.
\]

We now apply the results of section 3 to the electric field \(\mathcal{E}\). Although relations (14)–(19) are completely general, their practical utility is mainly limited to the case where \(\mathcal{F}(t)\) is a genuine finite-bandwidth, bandpass signal around \(\pm \omega_0\); i.e., one of the form (21) with \(\epsilon(t; z)\) being zero or of a magnitude significantly smaller than \(|\mathcal{E}^{+}(t; z)|, |\mathcal{E}^{-}(t; z)|\). For the sake of clarity in the presentation, we shall first assume \(\epsilon(t; z) \approx 0\). The general case \(\epsilon(t; z) \neq 0\) will be dealt with separately. We have, from (16),

\[
\tilde{\mathcal{E}}(t; z) = 2\mathcal{E}^{+}(t; z) \approx 2\mathcal{E}^{(+))(t; z)} = \frac{1}{2\pi} \int_{0}^{\infty} G(\omega - \omega_0) e^{i\omega t} e^{-i\beta(\omega) z} \, d\omega
\]

\[
= e^{i\omega t} e^{-i\beta(\omega_0) z} \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega) z} \, d\Omega,
\]

(23)

where the replacement \(\int_{0}^{\infty} = \int_{-\infty}^{\infty}\) is made consistently with the assumption \(e^{(+))(t; z)} = 0\.

Therefore,

\[
r(z; t) = e^{-i\omega_0 t} \tilde{\mathcal{E}}(t; z) = e^{-i\beta(\omega_0) z} \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega) z} \, d\Omega.
\]

(24)

The real envelope is thus

\[
r(z; t) = \left| e^{-i\beta(\omega_0) z} \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega) z} \, d\Omega \right|
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega) z} \, d\Omega
\]

\[
= \left| \text{FT}^{-1}\left\{ G(\Omega) e^{-i\Delta\beta(\Omega) z} \right\} \right|,
\]

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and

\[ E(t; z) = \left| \text{FT}^{-1} \left\{ G(\Omega) e^{-i\Delta\beta(\Omega)z} \right\} \right| \cos[\omega_0 t + \alpha(z; t)]. \quad (25) \]

We have thus proved that the received pulse envelope — the only magnitude obtained by direct optical detection — is always given by the modulus of the inverse Fourier transform of \( G(\Omega) \exp(-i\Delta\beta(\Omega)z) \). All other effects are encompassed by the phase term \( \alpha(z; t) \), to which the optical detector is blind. From (23), \( \alpha(z; t) \) is of the form

\[ \alpha(z; t) = -\beta(\omega_0)z + \varphi(z; t). \quad (26) \]

So \(-\beta(\omega_0)z\) always appears within the argument of the cosine, as expected, and

\[ \varphi(z; t) \equiv \arg \left[ \text{FT}^{-1} \left\{ G(\Omega) e^{-i\Delta\beta(\Omega)z} \right\} \right] = \arg \left\{ \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega)z} d\Omega \right\} \quad (27) \]

is identified with

\[ \frac{\beta z (t - z/v_g)^2}{2(T^4 + \beta_2 z^2)} - \frac{1}{2} \text{FT}^{-1} \left( \frac{\beta z}{T^2} \right) \]

in (5).

Consequently, we see that no extra-oscillations of the type (erroneously) found in [3] will arise in the photocurrent; not only for Gaussian pulses but for any pulse shape. Result (25) is totally general provided the approximate equalities in (23) hold.

We shall now repeat the precedent derivation with \( \epsilon(t; z) \neq 0 \). This is indeed an extreme case, as \(|\epsilon(t; z)|\) has a negligible magnitude even for the shortest pulses used in optical communications today. However, more generality and further insight will be gained with a full derivation that does not makes use of the approximation \( E^+(t; z) \simeq E^+(t; z) \).

We now write the exact expression of \( \tilde{E}(t; z) \):

\[ \tilde{E}(t; z) = 2E^+(t; z) = 2E^+(t; z) + 2\epsilon^+(t; z) \]

\[ = e^{i\omega_0 t e^{-i\beta(\omega_0)z}} \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega)z} d\Omega - 2\epsilon^-(t; z) + 2\epsilon^+(t; z). \quad (28) \]

The derivation of the last equality is straightforward using the definitions (22). Moreover,
\[-2e^{(-)}(t; z) + 2e^{(+)}(t; z) \equiv i\Delta\epsilon(t; z),
\]
\[
= \frac{i}{\pi} \int_0^\infty d\omega |G(\omega + \omega_0)| \sin \{\omega t - \beta(\omega)z + \gamma (\omega + \omega_0)\},
\]

(29)

which can be easily obtained taking into account that the FT of a real function \(g(t), G(\omega) = |G(\omega)| \exp[i\gamma(\omega)],\)
is even in modulus and odd in phase. From (29), we see that \(\Delta\epsilon(t; z)\) is a real quantity, a fact which will be used to compute \(r(z;t)\) below.

Therefore,

\[
\tilde{E}(t; z) = e^{i\omega_0 t} e^{-i\beta(\omega_0)z} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega)z} d\Omega + i\Delta\epsilon(t; z),
\]

and the complex envelope is

\[
r(z; t) = e^{-i\omega_0 t} \tilde{E}(t; z) = e^{-i\beta(\omega_0)z} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega)z} d\Omega + i\Delta\epsilon(t; z)e^{i\omega_0 t}
\]

\[
\equiv e^{-i\beta(\omega_0)z} B(z; t) + i\Delta\epsilon(t; z)e^{i\omega_0 t},
\]

where

\[
B(z; t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega G(\Omega) e^{i\Omega t} e^{-i\Delta\beta(\Omega)z} d\Omega = \text{FT}^{-1}\left\{G(\Omega) e^{-i\Delta\beta(\Omega)z}\right\};
\]

Note that \(B(z; t)\) coincides with the Fourier integral of the complex envelope (24) obtained in the simplified case, \(\Delta\epsilon(t; z) = 0\).

The real envelope is now calculated as

\[
r^2(z; t) = \left| r(z; t) \right|^2 = r(z; t)r^*(z; t)
\]

\[
= |B(z; t)|^2 + \left|\Delta\epsilon(t; z)\right|^2 - 2 |B(z; t)| \Delta\epsilon(t; z) \cos \left\{\omega_0 t + \beta_0 z - \varphi(z; t)\right\},
\]

with \(\varphi(z; t)\) defined as in (27). So we finally arrive at

\[
E(t; z) = |B(z; t)| \left(1 + \frac{\left|\Delta\epsilon(t; z)\right|^2}{|B(z; t)|^2} - 2 \frac{\Delta\epsilon(t; z)}{|B(z; t)|} \cos \left\{\omega_0 t + \beta_0 z - \varphi(z; t)\right\}\right)^{1/2}
\]

\[
\times \cos[\omega_0 t + \alpha(z; t)],
\]

(30)
with \( \alpha(z, t) = \arg \left[ r(z; t) \right] \), given by

\[
\tan \alpha(z, t) = \frac{|B(z; t)| \sin[\varphi(z; t) - \beta_0 z] + \Delta \epsilon(t; z) \cos(\omega_0 t)}{|B(z; t)| \cos[\varphi(z; t) - \beta_0 z] - \Delta \epsilon(t; z) \sin(\omega_0 t)}.
\]  \(31\)

Naturally, (30) reduces to (25) in the usual case \(|\Delta \epsilon(t; z)| \ll |B(z; t)|\). Likewise, (31) reduces to (26), as expected. Note that, when the effect of \(\Delta \epsilon(t; z)\) is taken into account, an (extremely small) “ringing” at the optical frequency \(\omega_0\) (and harmonics) appears in the otherwise “slow envelope” \(r(z; t)\). We can safely approximate, from (30),

\[
r(z; t) \simeq |B(z; t)| \left( 1 + \frac{[\Delta \epsilon(t; z)]^2}{2|B(z; t)|^2} - \frac{\Delta \epsilon(t; z)}{|B(z; t)|} \cos \{\omega_0 t + \beta_0 z - \varphi(z; t)\} \right).
\]

The second term in the parentheses is just a small correction to \(|B(z; t)|\). From (29), we see that \(\Delta \epsilon(t; z)\) essentially contains low (non-optical) frequencies, and so does \(B(z; t)\); consequently, \([\Delta \epsilon(t; z)]^2 / \left[ 2 |B(z; t)|^2 \right] \) will be detected. The third term containing \(\cos \{\omega_0 t + \beta_0 z - \varphi(z; t)\}\) has optical frequencies, so it would not be detected were it not for the beat frequencies that arise due to its product with \(\cos[\omega_0 t + \alpha(z, t)]\).

5 Concluding remarks

We have discussed a remarkable mistake found in the literature of dispersive pulse propagation in fibers (which has never pointed out by any author, to our knowledge). Using the previous discussion as a starting point, we have proved a novel, general result that provides solid grounds for the use of numerical techniques in linear pulse propagation problems. Expressions (25) and (26) represent the key results of this discussion. The even more general results (30) and (31), in which the originally slow envelope and phase are “contaminated” by optical frequency contributions, seem to be of less practical interest. However, useful insight can be drawn from them in assessing the behavior of extremely short pulses for which the narrow-band approximation is, in principle, arguable. \(^1\)

To get a quick idea of the orders of magnitude involved, consider, say, a Gaussian pulse \(\exp(-t^2/2T)\), with \(T \sim 10\) fs, over an optical carrier at \(1.3\) \(\mu\)m (\(\omega_0 = 1.45 \times 10^{15}\) rad/s); this means only \(\sim 12\) periods of optical carrier within the pulse envelope. Consider, for simplicity, \(z = 0\). In this case

\(^1\)In fact, at this point other effects would arise which could not be ignored: First, it would be difficult to establish an initial “modal profile”. \([6]\), \([7]\] Second, the pulse spectrum would spread as much as to cover other transmission windows, so losses would become very frequency-dependent.
\[
\frac{\Delta e(t = 0)}{B(t = 0)} = \frac{1}{\pi} \int_{0}^{\infty} d\omega |G(\omega + \omega_0)| \sin \{\gamma(\omega + \omega_0)\}
= \frac{1}{\pi} \int_{0}^{\infty} d\omega \sqrt{2\pi T} \exp(-T^2(\omega + \omega_0)^2/2) \simeq 1.2 \times 10^{-47} (!)
\]

The approximation \( \mathcal{E}^+(t; z) \simeq \mathcal{E}^{(+)}(t; z) \) is thus excellent even for ultrashort pulses.

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References


FIGURE CAPTIONS

Figure 1. Photocurrent corresponding to a directly-detected Gaussian pulse. The apparent ringing is merely an artifact arising from a conceptual mistake; the true detected pulse shape is that shown with a thin line.

Figure 2. Contributions neglected in the “narrow-band” approximation.